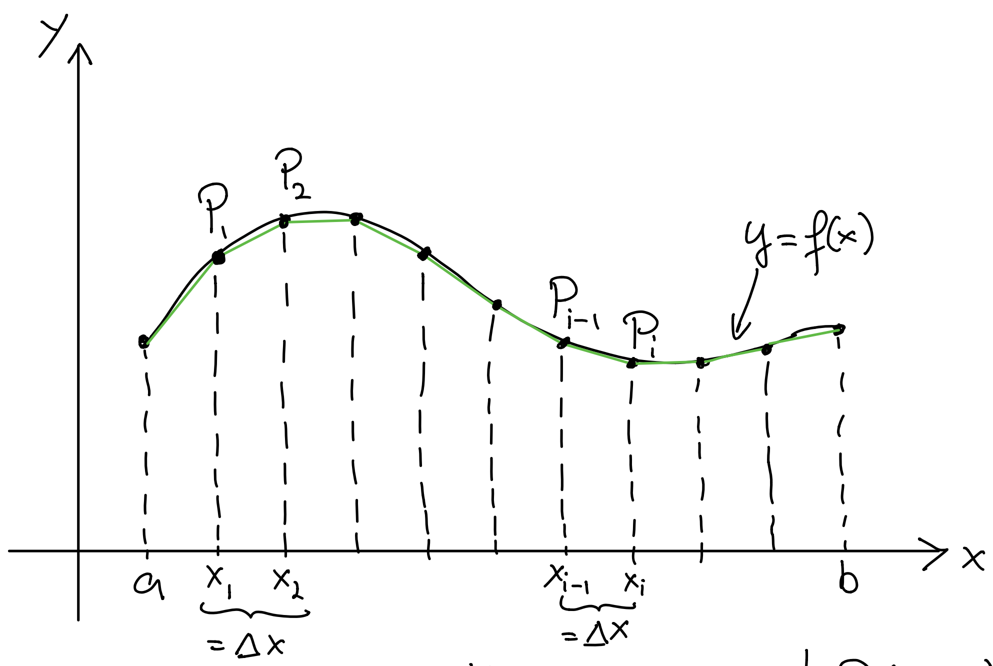
§ 8. Arc length, Surface of Revolution, and Polar coordinates

§ 8.1 Arc length:

What do we mean by the length of a curve?
Suppose a curve $\mathcal{C}$ is defined by the equation

$$y = f(x), \quad f \text{ continuous and } a \le x \le b$$

We obtain a polygonal approximation to $\mathcal{C}$ by dividing the interval $[a,b]$ into n subintervals:



If $y_i = f(x_i)$, then then the point $P_i(x_i, y_i)$ lies on $\mathcal{C}$ and the polygon with vertices

$P_0, P_1, \dots, P_n$ is an approximation to $\mathcal{C}$.

<u>Definition 8.1</u> ("length")

The "length" of the curve is given by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

We can derive an integral formula for L in case f has a continuous derivative, i.e. is "smooth" $(f \in C'((a,b)))$:

$\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1} P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Mean value Theorem

$$\Rightarrow \exists\ x_i^* \in (x_{i-1}, x_i) \text{ s.t.}$$

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is $\quad \Delta y_i = f'(x_i^*) \Delta x$

Thus we have

$$|P_{i-1} P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*) \Delta x]^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\ \sqrt{(\Delta x)^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \quad (\text{since } \Delta x > 0)$$

Therefore, by Definition 8.1,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$

We recognize this expression (by Th. 7.1) as

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

The integral exists because $g(x) = \sqrt{1+[f'(x)]^2}$ is continuous. Thus we have proved the following:

<u>Proposition 8.1:</u>

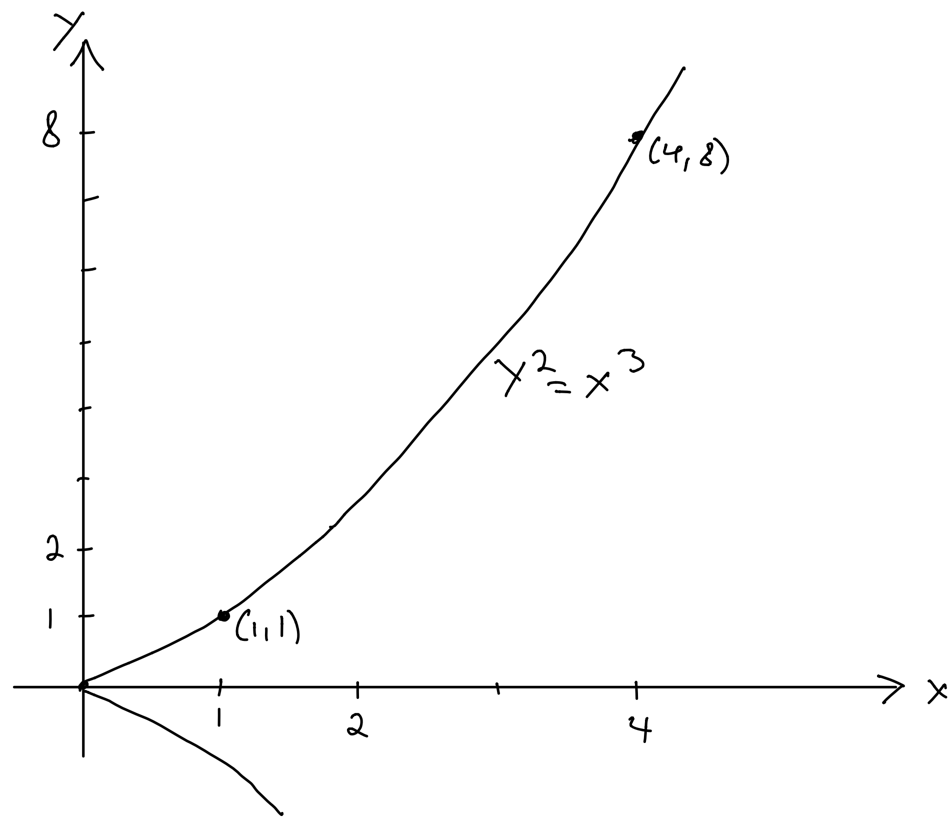
If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

<u>Example 8.1:</u>

Find the length of the arc of the semicubical $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$:



solution:

For the top half of the curve we have

$$y = x^{3/2} \quad , \quad \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

and so

$$L = \int_1^y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4} dx$

$\Rightarrow$ $x = 1$, $u = \frac{13}{4}$ and $x = 4$, $u = 10$ and

$$L = \frac{4}{9} \int_{13/4}^{10} \sqrt{u}\, du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

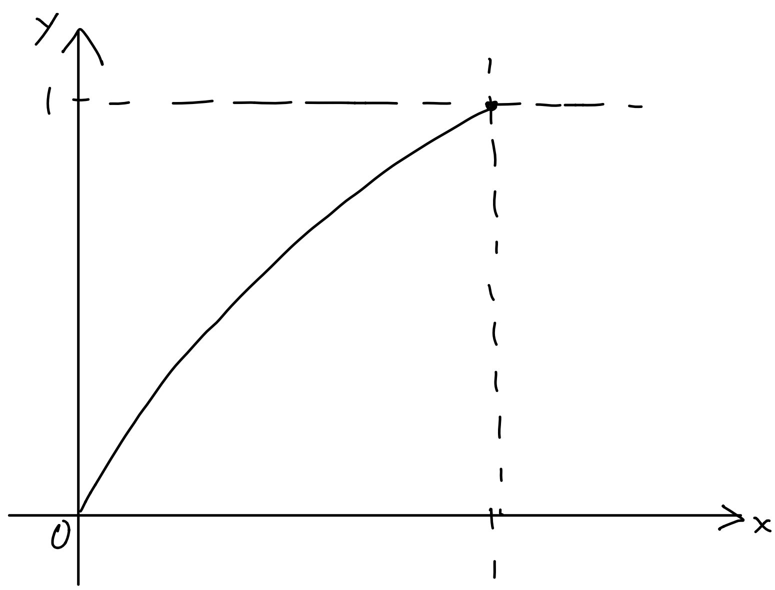
Remark 8.1:

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, we have

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 8.2:

Find the length of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.



Solution:

Since $x = y^2$, we have $\frac{dx}{dy} = 2y$ and thus

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, giving $dy = \frac{1}{2}\sec^2\theta\, d\theta$, and

$$\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sec\theta$$

When $y=0$, $\tan\theta = 0$, so $\theta = 0$; when $y=1$, $\tan\theta = 2$, so $\tan^{-1} 2 = \alpha$. Thus

$$L = \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|\Big]_0^\alpha$$

(leave as Homework)

$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \log|\sec\alpha + \tan\alpha|\right)$$

Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\log(\sqrt{5}+2)}{4}$$

<u>Definition 8.2</u> (arc length function):

Let C be a smooth curve given by $y = f(x)$, $a \le x \le b$, then the "arc length function" is:

$$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

We have

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Sometimes we write this in the form

$$(ds)^2 = (dx)^2 + (dy)^2$$

Example 8.3:

Find the arc length function for the curve $y = x^2 - \frac{1}{8}\log x$ taking $P_0(1,1)$ as the starting point.

Solution:

If $f(x) = x^2 - \frac{1}{8}\log x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

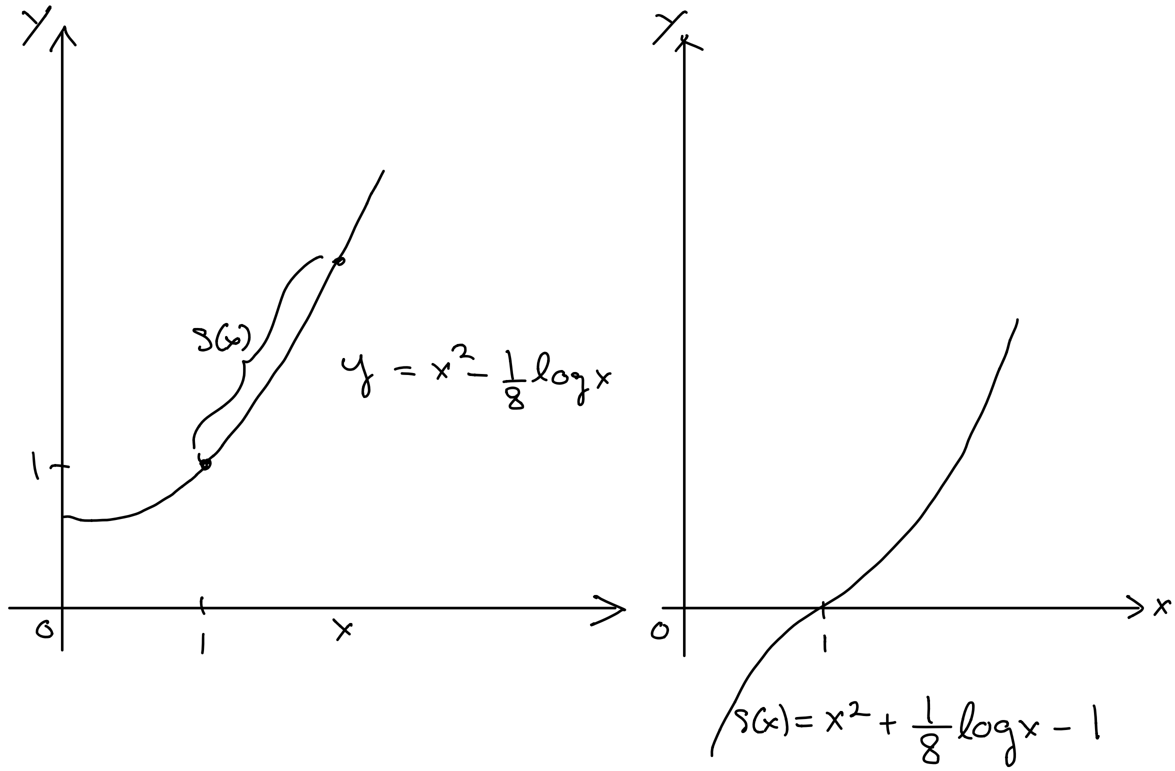
$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\Longrightarrow \sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x}$$

Thus the arc length is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \frac{1}{8}\log t \Big|_1^x = x^2 + \frac{1}{8}\log x - 1$$



y
1
S(x)
$y = x^2 - \frac{1}{8}\log x$
0
1
x
y
0
1
x
$S(x) = x^2 + \frac{1}{8}\log x - 1$